ON THE FORMAL AFFINE HECKE ALGEBRAS OF NORMAL FORMAL GROUP LAWS

CHANGLONG ZHONG

Abstract. We define the concept of normal formal group laws and study their properties. As an application, we simplify the notations of formal affine Demazure algebras and formal affine Hecke algebras. Moreover, for root systems of type different from G_2, we prove that the coefficients appeared in the relations of the generators of the formal affine Hecke algebras belong to the formal group algebra.

1. Introduction

Let R be a commutative ring and F be a formal group law over R. For the weight lattice Λ of a root system, the formal group algebra $R[[\Lambda]]_F$ was defined in [CPZ]. It can be thought as an algebraic replacement of the equivariant oriented cohomology theory of algebraic varieties. For each root α, the formal Demazure operator Δ_α^F acts on $R[[\Lambda]]_F$. Furthermore, generalizing the construction of nil Hecke algebras of [KK] (and some other literatures, for example, [Gin] and [Lus]), in [HMSZ], these operators were constructed as elements (called the formal Demazure elements $\mathit{\Delta}_\alpha^F$) in the semidirect product of $R[[\Lambda]]_F$ and the group ring $R[W]$ of the Weyl group W. The algebra generated by the formal Demazure elements $\mathit{\Delta}_\alpha$ is called the formal affine Demazure algebra $\mathbf{D}_F$. This provides a more convenient way to study the equivariant cohomology theory (for example, see [CZZ]).

On the other hand, in the classical theory of Hecke algebras (in which case $F = F_m$, the multiplicative formal group law), the Demzaure element $\mathit{\Delta}_\alpha^{F_m}$ can be thought as degeneration of generator $T_{s_\alpha} = T_{s_\alpha}^{F_m}$ of (affine) Hecke algebras. Following this direction, for arbitrary F, the formal affine Hecke algebra $\mathbf{H}_F$ was defined in [HMSZ]. It is generated by $T_{s_{\alpha_i}}^F$ with α_i among a fixed set of simple roots. The quadratic relations and the braid type relations between the generators were constructed, but the latter ones come with some auxiliary coefficients. As a result, $\mathbf{H}_F$ could only be considered as an algebra over some ring extended from the formal group algebra. The goal of this paper is to study these coefficients and to remove this extension.

For a (one-dimensional commutative) formal group law

$$F(x,y) = x + y + \sum_{i,j\geq 1} a_{ij}x^iy^j, \quad a_{ij} \in R$$

such that $a := a_{11}$ is invertible in R, we define a new formal group law $\tilde{F}$ and call it the associated normalization. This new formal group law is isomorphic to F but with more explicit and simple formal inverse element. With the normalization, we simplify the notations of formal (affine) Demazure algebras $\mathbf{D}_F$ and formal (affine)

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Hecke algebras $\mathbf{H}_F$. Moreover, we show that (Theorem 4.5), when a is invertible, the coefficients appeared in the braid type relations of the generators $T^F_{s_{\alpha_i}}$ belong to $R[[\Lambda]]_F$ for root systems of all types (except for type G_2). Consequently, $\mathbf{H}_F$ is an algebra over the formal group algebra. This fact is proved in [CZZ] but our proof makes the relations more explicit.

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Notations

Let R be an integral domain. Let (Λ, Φ, ρ) be a reduced root system, i.e., a free abelian group Λ of rank n (the weight lattice), a finite subset Φ of Λ whose elements are called the roots, and a map $\rho : \Lambda \to \Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}), \lambda \mapsto \lambda^\vee$ satisfying certain axioms. The reflection $\lambda \mapsto \lambda - (\alpha^\vee, \lambda)\alpha$ is denoted by s_α. Here $(\alpha^\vee, \lambda)$ is the canonical pairing between $\Lambda^\vee$ and Λ. The Weyl group is the group generated by $s_\alpha, \alpha \in \Phi$. We fix a set of simple roots $\{\alpha_1, ..., \alpha_n\} \subset \Phi$, and denote $[n] = \{1, ..., n\}$. Let ℓ be the length function, and let m_{ij} be the order of $s_i s_j$ in W for $i \neq j$. In this paper we consider root systems of all types except for type G_2. Therefore, $2 \le m_{ij} \le 4$

Let F be a one-dimensional formal group law over R, i.e., a power series

$$F(x, y) = x + y + \sum_{i,j \ge 1} a_{ij} x^i y^j, \quad a_{ij} \in R$$

satisfying the following properties

$$F(x, F(y, z)) = F(F(x, y), z), \quad F(x, y) = F(y, x), \quad \text{and} \quad F(x, 0) = x.$$

We sometimes write $x +_F y$ for $F(x, y)$, and denote the formal inverse of x by $\imath_F x$. In this paper we always assume that $a := a_{11} \neq 0$. For example, the polynomial $F_M(x, y) = x + y - axy$ with $a \neq 0$ defines the multiplicative formal group law.

2. Formal affine Demazure algebras and formal affine Hecke algebras

In the present section, we recall briefly the definitions and properties of formal group algebras [CPZ], formal affine Demazure algebras and formal affine Hecke algebras [HMSZ].

2.1 **Definition.** Consider the polynomial ring $R[x_\Lambda]$ in variables x_λ with $\lambda \in \Lambda$. Let

$$\epsilon : R[x_\Lambda] \to R, \quad x_\lambda \mapsto 0$$

be the augmentation map, and let $R[[x_\Lambda]]$ be the $(\ker \epsilon)$-adic completion of $R[x_\Lambda]$. Let $\mathcal{J}_F$ be the closure of the ideal of $R[[x_\Lambda]]$ generated by x_0 and elements of the form $x_{\lambda_1+\lambda_2} - F(x_{\lambda_1}, x_{\lambda_2})$ for all $\lambda_1, \lambda_2 \in \Lambda$. Here $x_0 \in R[x_\Lambda]$ is the element determined by the zero element of Λ. The *formal group algebra* $R[[\Lambda]]_F$ is defined to be the quotient

$$R[[\Lambda]]_F = R[[x_\Lambda]]/\mathcal{J}_F.$$

The augmentation map induces a ring homomorphism $\epsilon : R[[\Lambda]]_F \to R$, and we denote the kernel by $\mathcal{I}_F$. Then we have a filtration of ideals

$$R[[\Lambda]]_F = \mathcal{I}_F^0 \supseteq \mathcal{I}_F^1 \supseteq \mathcal{I}_F^2 \supseteq \cdots$$

and the associated graded ring

$$Gr_R(\Lambda, F) \overset{def}{=} \bigoplus_{i=0}^{\infty} \mathcal{I}_F^i / \mathcal{I}_F^{i+1}.$$

By [CPZ, Lemma 4.2], $Gr_R(\Lambda, F)$ is isomorphic to $S_R^*(\Lambda)$, the symmetric algebra. The isomorphism maps $\prod x_{\lambda_i}$ to $\prod \lambda_i$. Moreover, let $\{w_1, ..., w_n\}$ be the collection of fundamental weights, which is a basis of Λ, then $R[[\Lambda]]_F \cong R[[w_1, ..., w_n]]$. But the latter isomorphism is not canonical in general.

For any $x \neq 0$ in $R[[\Lambda]]_F$, define $\mu_F(x) = \frac{\imath_F x}{-x}$. For any root α, define $\kappa_\alpha^F = \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}}$. It is not difficult to see that κ_α^F and $\mu_F(x)$ belong to $R[[\Lambda]]_F$.

2.2 **Example.** For the multiplicative formal group law $F_M(x, y) = x + y - axy$, we have $\imath_M(x) = \frac{u}{au-1}$, $\mu_M(x) = \frac{1}{1-au}$ and $\kappa_\alpha^M = a$.

The action of the Weyl group W on Λ extends to an action on $R[[\Lambda]]_F$ by $s_\alpha(x_\lambda) = x_{s_\alpha(\lambda)}$. Define the following operator

$$\Delta_\alpha^F(x) = \frac{x - s_\alpha(x)}{x_\alpha}.$$

By [CPZ, Corollary 3.4], for any $x \in R[[\Lambda]]_F$, one has $\Delta_\alpha^F(x) \in R[[\Lambda]]_F$. In other words, it is an R-linear operator on $R[[\Lambda]]_F$, called the *Demazure operator*.

2.3 **Definition.** Let Q^F to be the field of fractions of $R[[\Lambda]]_F$. The action of W on $R[[\Lambda]]_F$ extends to an action on Q^F. Define the *twisted formal group algebra* to be the smash product $Q_W^F := R[W] \sharp Q^F$. In other words, Q_W^F is $R[W] \otimes_R Q^F$ as an R-module, and the multiplication is given by

$$q\delta_w \cdot q'\delta_{w'} = qw(q')\delta_{ww'} \quad \text{for } w, w' \in W \text{ and } q, q' \in Q^F.$$

Notice that Q_W^F is a free (left and right) Q^F-module with basis $\{\delta_w\}_{w \in W}$. But it is not a Q^F-algebra since the embedding $Q^F \to Q_W^F, q \mapsto q\delta_e$ is not central in Q_W^F. Here $e \in W$ is the identity in W. We denote δ_e by 1.

2.4 **Definition.** For each root $\alpha \in \Phi$, we define the *Demazure element*

$$\mathit{\Delta}_\alpha^F := \frac{1}{x_\alpha}(1 - \delta_{s_\alpha}) = \frac{1}{x_\alpha} - \delta_{s_\alpha}\frac{1}{x_{-\alpha}} \in Q_W^F.$$

Notice that $\mathit{\Delta}_\alpha^F$ is an element in Q_W^F while the Demazure operator Δ_α^F is an operator acting on $R[[\Lambda]]_F$. The difference of the notations shall help the reader to distinguish between them.

2.5 **Definition.** Define the *formal affine Demazure algebra* $\mathbf{D}_F$ to be the R-subalgebra of Q_W^F generated by $R[[\Lambda]]_F$ and $\mathit{\Delta}_{\alpha_i}$ for $i = 1, ..., n$.

If there is no confusion, we will denote $s_i = s_{\alpha_i}$, $x_i = x_{\alpha_i}$, $\kappa_i = \kappa_{\alpha_i}^F$, $\Delta_i = \Delta_{\alpha_i}^F$, $\mathit{\Delta}_i = \mathit{\Delta}_{\alpha_i}^F$ and $\delta_i = \delta_{s_i}$. We also denote $x_{\pm i} = x_{\pm\alpha_i}$, $x_{i\pm j} = x_{\alpha_i \pm \alpha_j}$. For any

$i \neq j \in [n]$, define

$$\kappa_{ij} = \kappa_{ij}^F = \frac{1}{x_{i+j}x_j} - \frac{1}{x_{i+j}x_{-i}} - \frac{1}{x_i x_j}, \tag{1}$$

$$\kappa'_{ij} = (\kappa_{ij}^F)' = \frac{1}{\kappa_{i+j}x_{i+j}\kappa_j x_j} - \frac{1}{\kappa_{i+j}x_{i+j}\kappa_i x_{-i}} - \frac{1}{\kappa_i x_i \kappa_j x_j}. \tag{2}$$

It is shown in [HMSZ] that $\kappa_{ij}^F \in R[[\Lambda]]_F$, and we will show that $\kappa'_{ij} \in R[\frac{1}{a}][[\Lambda]]_F$ in the last section.

If $I = (i_1, ..., i_l)$ is a sequence with $i_j \in [n]$, we sometimes write

$$\Delta_I = \Delta_{i_1 i_2 \ldots i_l} = \Delta_{i_1} \circ \cdots \Delta_{i_l},$$

and similarly for Δ_I. For each $w \in W$ and a reduced decomposition $w = s_{i_1} \cdots s_{i_r}$, define $\Delta_w = \Delta_{i_1} \cdots \Delta_{i_r}$ and $\Delta_w = \Delta_{i_1} \circ \cdots \Delta_{i_r}$. These definitions depend on the choice of the reduced decomposition of w unless F is the additive formal group law or the multiplicative formal group law [CPZ, Theorem 3.9].

prop:demazure

2.6 **Proposition.** [HMSZ, §5] *The algebra $\mathbf{D}_F$ satisfies the following properties:*

1. *For any $q \in Q^F$, we have $q\Delta_i = \Delta_i s_i(q) + \Delta_i(q)$.*
2. *For any $i \in [n]$, we have $\Delta_i^2 = \Delta_i \kappa_i$.*
3. *For any $i \neq j$, if $(\alpha_i^\vee, \alpha_j) = 0$, so that $m_{ij} = 2$, then $\Delta_{ij} = \Delta_{ji}$.*
4. *For any $i \neq j$, if $(\alpha_i^\vee, \alpha_j) = (\alpha_j^\vee, \alpha_i) = -1$, so that $m_{ij} = 3$, then*
$$\Delta_{jij} - \Delta_{iji} = \Delta_i \kappa_{ij} - \Delta_j \kappa_{ji}.$$
5. *For any $i \neq j$, if $(\alpha_i^\vee, \alpha_j) = -1$ and $(\alpha_j^\vee, \alpha_i) = -2$, so that $m_{ij} = 4$, then*
$$\begin{aligned}\Delta_{jiji} - \Delta_{ijij} &= \Delta_{ij}(\kappa_{i+2j,-j} + \kappa_{ji}) - \Delta_{ji}(\kappa_{i+j,j} + \kappa_{ij}) + \\ &\quad \Delta_j \cdot \Delta_i(\kappa_{i+j,j} + \kappa_{ij}) - \Delta_i \cdot \Delta_j(\kappa_{i+2j,-j} + \kappa_{ji}).\end{aligned}$$
6. *For each $w \in W$, choose a reduced decomposition of w and define Δ_w correspondingly. Then $\{\Delta_w\}_{w\in W}$ is a basis of $\mathbf{D}_F$ as a left (or right) $R[[\Lambda]]_F$-module.*
7. *As an R-algebra, $\mathbf{D}_F$ is generated by $R[[\Lambda]]_F$ and $\Delta_i, i \in [n]$ with relations (1) – (5) above.*

2.7 **Definition.** Fix a free abelian group Γ of rank 1 and denote $R_F = R[[\Gamma]]_F$. Let γ be a generator of Γ and denote by x_γ the corresponding element in R_F. Define $\Theta_F = \mu_F(x_\gamma) - \mu_F(x_{-\gamma})$.

Let $(Q^F)'$ be the fraction field of $R_F[[\Lambda]]_F$. Let $(Q_W^F)'$ be the respective twisted formal group algebra defined over R_F, i.e., $(Q_W^F)' = R_F[W]\sharp_{R_F}(Q^F)'$. For any $i \in [n]$, define the element

$$T_i^F := \Delta_i^F \frac{\Theta_F}{\kappa_i^F} + \delta_i \mu_F(x_\gamma).$$

Define the *formal affine Hecke algebra* $\mathbf{H}_F$ to be the R_F-subalgebra of $(Q_W^F)'$ generated by T_i^F, $i \in [n]$ and $R_F[[\Lambda]]_F$.

Notice that Θ_F, $\mu_F(x_\gamma)$ and κ_i^F are invariant under s_i, so they commute with Δ_i^F and δ_i. In other words, in the definition of T_i^F, it does not matter whether we write the coefficients $\frac{\Theta_F}{\kappa_i^F}$ and $\mu_F(x_\gamma)$ on the left or on the right.

If F is clear in the context, we write $T_i = T_i^F$, $\mu(x_\gamma) = \mu_F(x_\gamma)$ and $\Theta = \Theta_F$. For a sequence $I = (i_1, ..., i_l)$ with $i_j \in [n]$, we denote

$$T_I = T_{i_1 \dots i_l} = T_{i_1} \cdot \dots \cdot T_{i_l}.$$

For each $w \in W$, choose a reduced decomposition $w = s_{i_1} \cdots s_{i_r}$ and define $T_w = T_{i_1} \cdots T_{i_r}$. Unless F is the multiplicative or additive formal group law, this T_w will depend on the choice of reduced decomposition.

prop:hecke

2.8 **Proposition.** [HMSZ, §7] *For each $w \in W$, fix a reduced decomposition and define T_w correspondingly. The algebra $\mathbf{H}_F$ satisfies the following properties:*

(1) *For any $q \in (Q^F)'$ and $i \in [n]$, we have $qT_i - T_i s_i(q) = \frac{\Theta}{\kappa_i}\Delta_i(q)$.*
(2) *For any $i \in [n]$, we have $T_i^2 = T_i\Theta + 1$.*
(3) *For any $i \neq j$, we have*

$$\underbrace{T_jT_iT_j\cdots}_{m_{ij}\ terms} - \underbrace{T_iT_jT_i\cdots}_{m_{ij}\ terms} = \sum_{w\in W, \ell(w)\le m_{ij}-2} T_w\tau_w^{ij}$$

for some $\tau_{ij}^w \in (Q^F)'$.
(4) *For any $i \neq j$, if $(\alpha_i^\vee, \alpha_j) = 0$, so that $m_{ij} = 2$, then $T_{ij} = T_{ji}$.*
(5) *for any $i \neq j$, if $(\alpha_i^\vee, \alpha_j) = (\alpha_j^\vee, \alpha_i) = -1$, so that $m_{ij} = 3$, then*

$$T_{jij} - T_{iji} = \Theta^2(T_i - T_j)\kappa'_{ij}.$$

Moreover, $\kappa'_{ij} = \kappa'_{ji}$.

Define $R_F[[\Lambda]]_F^\sim = R_F[[\Lambda]]_F[\tau_w^{ij} | i,j \in I, w \in W, \ell(w) \le m_{ij}-2]$. Then:

(6) *Then the set $\{T_w\}_{w\in W}$ forms a basis of $\mathbf{H}_F \otimes_{R_F[[\Lambda]]_F} R_F[[\Lambda]]_F^\sim$ as a right $R_F[[\Lambda]]_F^\sim$-module.*
(7) *As an R_F-algebra, $\mathbf{H}_F \otimes_{R_F[[\Lambda]]_F} R_F[[\Lambda]]_F^\sim$ is generated by $R_F[[\Lambda]]_F^\sim$ and T_i with relations (1) – (5) above.*

3. Normalization of formal group laws

In this section we will define the concept of a normal formal group law. Recall that $\imath_F x$ is the formal inverse of x, i.e., $x +_F \imath_F x = 0$. Then it is easy to show that

$$\imath_F x = -x + ax^2 - a^2x^2 + \dots,$$

where $a = a_{11}$ is the coefficient of xy in $F(x, y)$. Define the power series

def:h(x)

$$h(x) = \frac{\imath_F x + x}{x} \tag{3}$$

so that $\imath_F x = -x + xh(x)$. The power series $h(x)$ has leading term ax, and if a is invertible in R, so h has a composition inverse. That is, there is a power series $f(x) \in xR[[x]]$ such that

$$f(h(x)) = h(f(x)) = x.$$

Recall that a homomorphism of formal group laws $g : F \to G$ is a power series $g(x) \in R[[x]]$ such that

$$g(x) +_G g(y) = g(x +_F y).$$

thm:fglkey

3.1 **Theorem.** *If a is invertible in R, then there exists a formal group law $\tilde{F}$ such that $h : F \to \tilde{F}$ is an isomorphism with the inverse $f : \tilde{F} \to F$. Moreover, $\imath_{\tilde{F}} x = \frac{x}{x-1}$, $\frac{1}{x} + \frac{1}{\imath_{\tilde{F}} x} = 1$ and $\mu_{\tilde{F}}(x) = \frac{1}{1-x}$.*

Proof. Define

$$x +_{\tilde{F}} y = h(f(x) +_F f(y)).$$

It is straightforward to show that $\tilde{F}$ is a formal group law. By definition,

$$h(x) +_{\tilde{F}} h(y) = h(f(h(x)) +_F f(h(y))) = h(x +_F y).$$

So $h : F \to \tilde{F}$ defines a homomorphism of formal group laws. Since $f(h(x)) = h(f(x)) = x$, so h is an isomorphism and $f : \tilde{F} \to F$ is its inverse.

Next we compute the inverse element with respect to $\tilde{F}$. Let $y = h(\imath_F f(x))$, then

$$x +_{\tilde{F}} y = h(f(x) +_F f(h(\imath_F f(x)))) = h(f(x) +_F \imath_F f(x)) = 0.$$

Recall that by definition, $\imath_F x = (h(x) - 1)x$, therefore, by uniqueness of inverse, we see that

$$\imath_{\tilde{F}} x = y = h(\imath_F f(x)) = h(-f(x) + f(x)h(f(x))) = h((x-1)f(x)).$$

On the other hand, we have

$$\begin{aligned} 1 &= \frac{\imath_F x}{-x} \frac{x}{-\imath_F x} \\ &= (1 - h(x))(1 - h(\imath_F x)) \\ &= (1 - h(x))(1 - h((h(x) - 1)x)) \end{aligned}$$

Letting $h(x) = z$, then $x = f(z)$, we see that

$$1 = (1 - z)(1 - h((z - 1)f(z))) = (1 - z)(1 - \imath_{\tilde{F}} z).$$

Therefore, $\imath_{\tilde{F}} z = \frac{z}{z-1}$. Hence, $\frac{1}{x} + \frac{1}{\imath_{\tilde{F}} x} = 1$ and $\mu_{\tilde{F}}(x) = \frac{\imath_{\tilde{F}}(x)}{-x} = \frac{1}{1-x}$. □

3.2 **Definition.** Suppose that $a \neq 0$. We say that a formal group law F is *normal* if $\imath_F(x) = \frac{x}{x-1}$. If F is not normal but a is invertible in R, then the associated normal formal group law $\tilde{F}$ exists, and we call it the *normalization* of F.

3.3 **Example.** For multiplicative formal group law $F_M(x, y) = x + y - axy$ with a invertible in R, we have $\imath_{F_M} u = -u + uh_M(u)$ with $h_M(u) = \frac{au}{au-1}$. So its composition inverse is $f_M(u) = \frac{1}{a}\frac{1}{1-u}$. Therefore,

$$\tilde{F}_M(x, y) = h_M(f_M(x) +_{F_M} f_M(y)) = x + y - xy.$$

This justifies the name "normalization".

3.4 **Lemma.** *If F is normal, then $h(x) = \imath_F(x)$. In particular, the isomorphism $h : F \to \tilde{F}$ maps x to $\imath_F(x)$.*

Proof. Since F is normal, $\imath_F(x) = \frac{x}{x-1}$. Therefore,

$$h(x) = \frac{\imath_F(x) + x}{x} = \frac{\frac{x}{x-1} + x}{x} = \frac{x}{x-1} = \imath_F(x).$$

□

4. FORMAL AFFINE DEMAZURE ALGEBRAS AND FORMAL AFFINE HECKE ALGEBRAS OF NORMAL FORMAL GROUP LAWS

In the present section, we apply the normalization of formal group laws to simplify the notations of formal affine Demazure algebras $\mathbf{D}_F$ and formal affine Hecke algebras $\mathbf{H}_F$. As an application, we prove the main result (Theorem 4.5) of this paper.

Recall that $h(x) = \frac{\imath_F x + x}{x}$ and f is its composition inverse.

lemma:1

4.1 **Lemma.** *If a is invertible in R, then there is a canonical isomorphism of rings*

$$\phi_f : R[[\Lambda]]_F \to R[[\Lambda]]_{\tilde{F}}, \quad x_\lambda \mapsto f(x_\lambda)$$

with inverse

$$\phi_h : R[[\Lambda]]_{\tilde{F}} \to R[[\Lambda]]_F, \quad x_\lambda \mapsto h(x_\lambda).$$

In particular, there are isomorphisms $\phi_f : R_F[[\Lambda]]_F \to R_{\tilde{F}}[[\Lambda]]_{\tilde{F}}$ and $\phi_h : R_{\tilde{F}}[[\Lambda]]_{\tilde{F}} \to R_F[[\Lambda]]_F$ inverse to each other.

Proof. Define

$$\phi_h : R[[\Lambda]]_{\tilde{F}} \to R[[\Lambda]]_F, \quad x_\lambda \mapsto h(x_\lambda)$$

and

$$\phi_f : R[[\Lambda]]_F \to R[[\Lambda]]_{\tilde{F}}, \quad x_\lambda \mapsto f(x_\lambda)$$

and extend linearly. Since $f(h(x)) = x = h(f(x))$, by [CPZ, §2], they define isomorphisms between $R[[\Lambda]]_F$ and $R[[\Lambda]]_{\tilde{F}}$.

Similarly, since $R_F[[\Lambda]]_F = R[[\Gamma \oplus \Lambda]]_F$, the conclusion of the second part follows. □

lemma:2

4.2 **Lemma.** (1) *If F is normal, then $\kappa^F_\alpha = 1$ and*

eq:kappaij

$$\kappa^F_{ij} = \frac{1}{x_{i+j}x_i} + \frac{1}{x_{i+j}x_j} - \frac{1}{x_{i+j}} - \frac{1}{x_i x_j}. \tag{4}$$

Moreover, $\kappa^F_{ij} = \kappa^F_{ji} = \kappa^F_{-i,-j} = \kappa^F_{-i,i+j}$.

(2) *If a is invertible in R, then $(\kappa^F)'_{ij} \in R[[\Lambda]]_F$.*

Proof. (1) By Theorem 3.1, we see that $\kappa^F_\alpha = 1$. Therefore,

eq:kappa

$$\frac{1}{x_{-i}} = 1 - \frac{1}{x_i}. \tag{5}$$

Substituting it into κ^F_{ij}, we get the formula (4). Since the formula (4) is symmetric with respect to i and j, so $\kappa^F_{ij} = \kappa^F_{ji}$. Moreover, using identity (5) we can verify that $\kappa^F_{ij} = \kappa^F_{-i,-j} = \kappa^F_{-i,i+j}$.

(2) By Lemma 4.1, there is an isomorphism $\phi_f : R_F[[\Lambda]]_F \to R_{\tilde{F}}[[\Lambda]]_{\tilde{F}}$, which extends to an isomorphism $\phi_f : (Q^F)' \to (Q^{\tilde{F}})'$. Consider $\phi_f((\kappa^F_{ij})')$. Since $x_i\kappa^F_i = h(x_{-i})$, so

$$\phi_f(x_i\kappa^F_i) = \phi_f(h(x_{-i})) = h(f(x_{-i})) = x_{-i} \in R[[\Lambda]]_{\tilde{F}}.$$

Therefore, $\phi_f((\kappa^F_{ij})')) = \kappa^{\tilde{F}}_{-i,-j} \in R[[\Lambda]]_{\tilde{F}}$. So $(\kappa^F_{ij})' = \phi_h(\kappa^{\tilde{F}}_{-i,-j}) \in R[[\Lambda]]_F$. □

lemma:3

4.3 **Lemma.** *If a is invertible in R, then the map ϕ_f induces ring isomorphisms $\phi_f : \mathbf{D}_F \to \mathbf{D}_{\tilde{F}}, \Delta^F_i \mapsto \Delta^{\tilde{F}}_i$ and $\imath_{\tilde{F}} \circ \phi_f : \mathbf{H}_F \to \mathbf{H}_{\tilde{F}}, T^F_i \mapsto T^{\tilde{F}}_i$.*

Proof. The map ϕ_f induces an isomorphism $\phi_f : Q_W^F \to Q_W^{\tilde{F}}, q\delta_w \mapsto \phi_f(q)\delta_w$. Moreover,

$$\phi_f(\Delta_i^F) = \frac{1}{f(x_i)} - \frac{1}{f(x_i)}\delta_i = \frac{x_i}{f(x_i)}\Delta_i^{\tilde{F}}.$$

Since $h(x)$ has leading term ax, so $f(x)$ has leading term $\frac{1}{a}x$, therefore, $\frac{x_i}{f(x_i)} \in R[[\Lambda]]_{\tilde{F}}$. Hence, ϕ_f induces a ring homomorphism $\phi_f : \mathbf{D}_F \to \mathbf{D}_{\tilde{F}}$. Similarly, ϕ_h induces $\phi_h : \mathbf{D}_{\tilde{F}} \to \mathbf{D}_F$ which is inverse to ϕ_f, hence $\mathbf{D}_F \cong \mathbf{D}_{\tilde{F}}$.

Recall that $x_i\kappa_i = h(x_{-i})$. So applying the isomorphism $\phi_f : R_F[[\Lambda]]_F \to R_{\tilde{F}}[[\Lambda]]_{\tilde{F}}$, by the proof of Lemma 4.2, we see that

$$\begin{aligned}
\phi_f(x_i\kappa_i^F) &= x_{-i},\\
\phi_f(\mu_F(x_\gamma)) &= \phi_f(1-h(x_\gamma)) = 1-h(f(x_\gamma)) = 1-x_\gamma = \mu_{\tilde{F}}(x_{-\gamma}),\\
\phi_f(\Theta_F) &= \mu_{\tilde{F}}(x_{-\gamma}) - \mu_{\tilde{F}}(x_\gamma).
\end{aligned}$$

Therefore, applying the isomorphism $\imath_{\tilde{F}} : R_{\tilde{F}}[[\Lambda]]_{\tilde{F}} \to R_{\tilde{F}}[[\Lambda]]_{\tilde{F}}$, we get that

$$\imath_{\tilde{F}} \circ \phi_f(x_i\kappa_i^F) = x_i, \quad \imath_{\tilde{F}} \circ \phi_f(\mu_F(x_\gamma)) = \mu_{\tilde{F}}(x_\gamma), \quad \imath_{\tilde{F}} \circ \phi_f(\Theta_F) = \Theta_{\tilde{F}}.$$

So the induced isomorphism $\imath_{\tilde{F}} \circ \phi_f : (Q_W^F)' \to (Q_W^{\tilde{F}})'$ maps

$$T_i^F = \Delta_i^F \frac{\Theta_F}{\kappa_i^F} + \delta_i\mu_F(x_\gamma) = \frac{\Theta_F}{x_i\kappa_i^F} + (\mu_F(x_\gamma) - \frac{\Theta_F}{x_i\kappa_i^F})\delta_i$$

to

$$T_i^{\tilde{F}} = \Delta_i^{\tilde{F}}\Theta_{\tilde{F}} + \mu_{\tilde{F}}(x_\gamma)\delta_i.$$

Clearly $\phi_h \circ \imath_{\tilde{F}}$ is its inverse. Therefore, $\mathbf{H}_F \cong \mathbf{H}_{\tilde{F}}$. ∎

Notice that the map $\phi_h \circ \imath_{\tilde{F}} : R_{\tilde{F}}[[\Lambda]]_{\tilde{F}} \to R_F[[\Lambda]]_F$ maps x_i to $x_i\kappa_i^F$, hence $\phi_h \circ \imath_{\tilde{F}}(\kappa_{ij}^{\tilde{F}}) = (\kappa_{ij}^F)'$.

We simply the notations of $\mathbf{D}_F$:

thm:demazure

4.4 **Theorem.** *If F is normal, the algebra $\mathbf{D}_F$ satisfies the following properties:*

(1) *For any $\alpha \in \Phi$, $\Delta_\alpha^2 = \Delta_\alpha$.*

(2) *If $(\alpha_i^\vee, \alpha_j) = (\alpha_j^\vee, \alpha_i) = -1$, so that $m_{ij} = 3$, then*

$$\Delta_{jij} - \Delta_{iji} = (\Delta_i - \Delta_j)\kappa_{ij}.$$

Moreover, κ_{ij} is invariant under s_i and s_j.

(3) *If $(\alpha_i^\vee, \alpha_j) = -1$ and $(\alpha_j^\vee, \alpha_i) = -2$, so that $m_{ij} = 4$, then*

$$\Delta_{jiji} - \Delta_{ijij} = (\Delta_{ij} - \Delta_{ji})(\kappa_{ij} + \kappa_{i,i+j}).$$

Moreover, $\kappa_{ij} + \kappa_{i,i+j}$ is invariant under s_i and s_j.

Proof. (1) By Lemma 4.2, $\kappa_\alpha = 1$. Therefore, from Proposition 2.6, we get the conclusion.

(2) By Lemma 4.2, we see that $\kappa_{ij} = \kappa_{ji}$. Moreover, direct computation shows that $s_i(\kappa_{ij}) = s_j(\kappa_{ij}) = \kappa_{ij}$. The conclusion then follows from Proposition 2.6.

(3) By Lemma 4.2, we see that $\kappa_{i+2j,-j} = \kappa_{-j,i+2j} = \kappa_{j,i+j}$. Moreover, direct computation shows that $\kappa_{ij} + \kappa_{i,i+j}$ is invariant under s_i and s_j. Therefore, $\Delta_i(\kappa_{ij} + \kappa_{i,i+j}) = \Delta_j(\kappa_{ij} + \kappa_{i,i+j}) = 0$. Then conclusion then follows from Proposition 2.6. ∎

thm:hecke

4.5 **Theorem.** *If F is normal, then the algebra $\mathbf{H}_F$ satisfies the following properties:*

(1) *If $(\alpha_i^\vee, \alpha_j) = (\alpha_j^\vee, \alpha_i) = -1$, so that $m_{ij} = 3$, then*

$$T_{jij} - T_{iji} = (T_i - T_j)\Theta_F^2 \kappa_{ij}^F.$$

(2) *If $(\alpha_i^\vee, \alpha_j) = -1$ and $(\alpha_j^\vee, \alpha_i) = -2$, so that $m_{ij} = 4$, then*

$$T_{jiji} - T_{ijij} = (T_{ij} - T_{ji})\Theta_F^2(\kappa_{ij}^F + \kappa_{j,i+j}^F).$$

(3) $R_F[[\Lambda]]_F = R_F[[\Lambda]]_F^{\sim}$.

(4) *For each $w \in W$, fix a reduced decomposition of w and define T_w correspondingly. The set $\{T_w\}_{w\in W}$ forms a basis of $\mathbf{H}_F$ as a right (or left) $R_F[[\Lambda]]_F$-module.*

(5) *As an R-algebra, $\mathbf{H}_F$ is generated by $R_F[[\Lambda]]_F$ and $T_i, i \in [n]$ with relations (1), (2) and (4) of Proposition 2.8 and (1) and (2) of this theorem.*

If F is not normal but a is invertible in R, then $\mathbf{H}_F$ satisfies the above properties after replacing κ_{ij}^F (resp. $\kappa_{j,i+j}^F$) by $(\kappa_{ij}^F)'$ (resp. $(\kappa_{j,i+j}^F)'$).

Proof. (1): By Lemma 4.2, $\kappa_\alpha^F = 1$ for any root α. The conclusion then follows from Proposition 2.8.

(2): By direct computation of $T_{jiji} - T_{ijij}$ based on the fact that $\kappa_\alpha = 1$, one gets the conclusion.

(3): Since we only consider root systems of type different from type G_2, so $m_{ij} \le 4$. Therefor, the elements τ_w^{ij} appeared in Proposition 2.8 belong to the set $\{\kappa_{ij}^F, \kappa_{ij}^F + \kappa_{j,i+j}^F\} \subset R_F[[\Lambda]]_F$. So $R_F[[\Lambda]]_F^{\sim} = R_F[[\Lambda]]_F$.

(4) and (5): These are direct consequence of (1)-(3) of this theorem and Proposition 2.8.

If F is not normal but a is invertible, then its normalization $\tilde{F}$ exists. By Lemma 4.3, there exist isomorphisms $\imath_{\tilde{F}} \circ \phi_f : R_F[[\Lambda]]_F \cong R_{\tilde{F}}[[\Lambda]]_{\tilde{F}}$ and $\imath_{\tilde{F}} \circ \phi_f : \mathbf{H}_F \cong \mathbf{H}_{\tilde{F}}$. The latter isomorphism maps T_i^F to $T_i^{\tilde{F}}$. By the proof of Lemma 4.2, the map $\imath_{\tilde{F}} \circ \phi_f$ maps $(\kappa_{ij}^F)'$ to $\kappa_{ij}^{\tilde{F}}$. Therefore, the R-algebra $\mathbf{H}_F$ satisfies the properties (1)-(5) above after replacing κ_{ij}^F by $(\kappa_{ij}^F)'$. □

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Changlong Zhong, Fields Institute and University of Ottawa

E-mail address: zhongusc@gmail.com